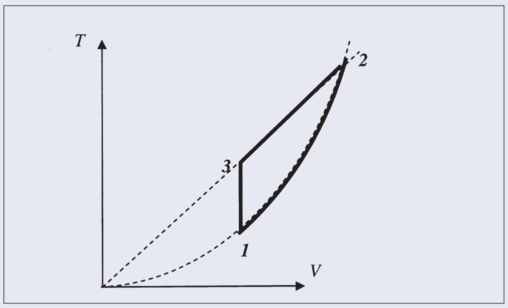
Physics Challenge for Teachers and Students

Solution to January 2008 Challenge

The Adventures of a Mole

Challenge: A mole of helium follows cycle 1-2-3-1 shown in the diagram. During process 1-2 the temperature of the gas depends on its volume as $T = bV^2$, where b is a constant. During the cycle, the maximum temperature of the gas is four times greater than the minimum temperature. If the gas absorbs the amount of heat Q during process 1-2, how much heat Q' does the gas dissipate during process 2-3-1?



Solution: Assume that each process occurs quasistatically so that all energy transfers are reversible, and take the helium to be ideal so that its pressure P, volume V, and temperature T are related by $PV = nRT$, where $n = 1$ mol, and $R = 8.314$ J/K/mol.

The problem can be readily solved if we transform the given T-V graph into a conventional P-V graph. During the $1 \rightarrow 2$ process for which $T = bV^2$, the pressure of the helium is given by

$$P = \frac{nRbV^2}{V} = nRbV, \tag{1}$$

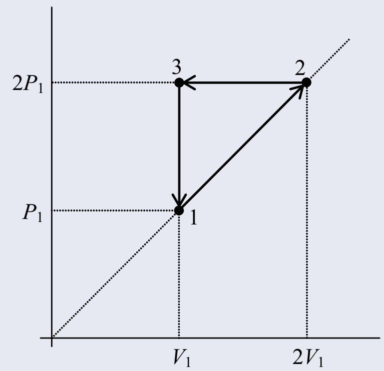
and thus is linearly proportional to the volume. Since $T_{max} = 4T_{min}$, we conclude that

$$T_2 = 4T_1 \quad \Rightarrow \quad bV_2^2 = 4bV_1^2 \quad \Rightarrow \quad V_2 = 2V_1 \quad \Rightarrow \quad P_2 = 2P_1. \tag{2}$$

Next, during the $2 \rightarrow 3$ process we see from the given graph that T is linearly proportional to V so that $T = aV$, say, where a is some constant. This relation implies that the $2 \rightarrow 3$ process is isobaric since $P = nRT/V = nRa$ is a constant along it, so that $P_3 = P_2 = 2P_1$. We also see from the given graph that $V_3 = V_1$ (i.e., the $3 \rightarrow 1$ process is isochoric). We thereby deduce that

$$V_2 = 2V_1 = 2V_3 \quad \Rightarrow \quad T_2/a = 2T_3/a \quad \Rightarrow \quad T_2 = 4T_1 = 2T_3. \tag{3}$$

We can summarize these results by drawing the following P-V graph.



Continuing, let's adopt the sign convention that work W and heat Q are positive when energy is added to the system. Then an infinitesimal amount of work done on the system is

$$dW = -PdV = -\frac{nRT}{V}dV. \tag{4}$$

Thus the work done on the helium during the $1 \rightarrow 2$ process is

$$W(1 \rightarrow 2) = -\int_{V_1}^{V_2} nRbV\,dV = -\frac{1}{2}nRb\left(V_2^2 - V_1^2\right) = -\frac{1}{2}nR\left(T_2 - T_1\right) = -\frac{3}{2}nRT_1, \tag{5}$$

using Eqs. (1) and (2). We are told that the heat transferred to the gas during this process is $Q(1\rightarrow 2) = Q$. Furthermore, the first law of thermodynamics states that a system gains infinitesimal internal energy dU when either an increment of work is done on it or a small bit of heat is transferred to it,

$$dU = dQ + dW \quad \Rightarrow \quad \Delta U = Q + W. \tag{6}$$

We therefore conclude that

$$\begin{aligned} \Delta U(1\rightarrow 2) &= W(1\rightarrow 2) + Q(1\rightarrow 2) \\ &= -\frac{3}{2}nRT_1 + Q. \end{aligned} \tag{7}$$

But we also know that for an ideal monatomic gas, $U = \frac{3}{2}nRT$, so that

$$\Delta U(1\rightarrow 2) = \frac{3}{2}nR\left(T_2 - T_1\right) = \frac{9}{2}nRT_1. \tag{8}$$

Equating the right-hand sides of Eqs. (7) and (8), we thus deduce that

$$T_1 = \frac{Q}{6nR}. \tag{9}$$

Next let's consider the two processes that return the gas to its initial state 1. The work done on the gas during the second process is

$$\begin{aligned} W(2\rightarrow 3) &= -\int_{V_2}^{V_3} nRa\, dV = -nRa\left(V_3 - V_2\right) \\ &= nR\left(T_2 - T_3\right) = 2nRT_1, \end{aligned} \tag{10}$$

using Eq. (3). Finally we note that the volume remains constant during the $3\rightarrow 1$ process so that $dV = 0$ and hence $W(3\rightarrow 1) = 0$. For one full cycle, we thus find that the work done on the system is

$$W_{\text{cycle}} = W(1\rightarrow 2) + W(2\rightarrow 3) + W(3\rightarrow 1) = \frac{1}{2}nRT_1, \tag{11}$$

using Eqs. (5) and (10). At the same time, the heat added to the gas during the cycle is

$$Q_{\text{cycle}} = Q(1\rightarrow 2) + Q(2\rightarrow 3\rightarrow 1) = Q - Q', \tag{12}$$

where the minus sign reflects the fact that we are told that Q' is dissipated to the surroundings, rather than being absorbed as Q is.

Now consider Eq. (6) again. Unlike Q and W, U is a function of state and hence when we take the system around a complete cycle and back to its initial state 1, its internal energy must return to its initial value and thus $\Delta U_{\text{cycle}} = 0$. We conclude that the sum of Eqs. (11) and (12) must be zero, so that

$$Q' = Q + \frac{1}{2}nRT_1 = \boxed{\frac{13}{12}Q}, \tag{13}$$

using Eq. (9) in the last step. Note that the net heat out, $Q' - Q$, is equal to the net work in, which is equal to the triangular area $\frac{1}{2}P_1V_1 = \frac{1}{2}nRT_1$ in the preceding P-V graph. We could have used this to directly compute Eq. (11) without having to perform the integrations in Eqs. (5) and (10), except that we needed Eq. (5) to derive Eq. (9).

One interesting addendum to this problem is that we can compute *numerical* values for the changes in entropy during each process. Since each step is reversible, we have

$$\begin{aligned} &dS \equiv \frac{dQ}{T} = \frac{dU}{T} - \frac{dW}{T} = \frac{3nRdT}{2T} + \frac{nRdV}{V} \quad \Rightarrow \\ &\Delta S = \frac{3}{2}nR\ln\left(\frac{T_f}{T_i}\right) + nR\ln\left(\frac{V_f}{V_i}\right), \end{aligned} \tag{14}$$

using Eqs. (4) and (6). For the $1\rightarrow 2$ process we use Eq. (2) to find

$$\begin{aligned} \Delta S\left(1\rightarrow 2\right) &= \frac{3}{2}nR\ln 4 + nR\ln 2 \\ &= 4nR\ln 2 = 23.05 \text{ J/K}. \end{aligned} \tag{15}$$

Next for the $2\rightarrow 3$ process we see from Eq. (3) that

$$\begin{aligned} \Delta S\left(2\rightarrow 3\right) &= \frac{3}{2}nR\ln\frac{1}{2} + nR\ln\frac{1}{2} = -\frac{5}{2}nR\ln 2 \\ &= -14.41 \text{ J/K}. \end{aligned} \tag{16}$$

Finally for the last step,

$$\Delta S\,(2 \to 3) = \frac{3}{2} nR \ln \frac{1}{2} + 0$$

$$= -\frac{3}{2} nR \ln 2 = -8.64 \text{ J/K.} \qquad (17)$$

As a check, note that if we add together these three entropy changes, we obtain $\Delta S_{cycle} = 0$ as we must since entropy is another state function.

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